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Influence of stochastic disturbances on the quasi-wavelet solution of the convection–diffusion equation*

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A quasi-wavelet numerical method (QWNM) is introduced for solving the convection–diffusion equation (CDE). The results manifest that the calculated bandwidth has an extremum. When the bandwidth takes the value of the extremum, the accuracy of the solution for the CDE by using the QWNM is relatively high, and better than that by using the up-wind scheme. Under the condition of stochastic boundary disturbances of different amplitudes, the results of the QWNM are a little worse than those of the up-wind scheme when the integral time is longer. However, when stochastic boundary disturbances of equal amplitudes occur, the solutions of the equation by using the QWNM and the up-wind scheme can be identical if the bandwidth takes an integer greater than or equal to 20. When the parameter is stochastically disturbed, the root-mean-square error of the quasi-wavelet solution of the equation is smaller than that of the up-wind scheme solution if the bandwidth is 10. When the initial values are stochastically disturbed and the bandwidth equals 10, the accuracy of the quasi-wavelet solution is relatively high, and better than that of the up-wind scheme solution.

Keywords: stochastic disturbance, convection–diffusion equation, quasi-wavelet, up-wind scheme
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1. Introduction

The convection–diffusion equation (CDE) is the simplest model of the Navier–Stokes equation, which describes the viscid incompressible flow, and is widely used in the scientific–technological fields, such as fluid mechanics, atmospheric dynamics, environmental protection, and chemical engineering: for example in relation to the motion and thermal conductivity of fluid (regulation of motion of ground water, petroleum, and natural gas), the transfer and diffusion of pollutants, electrochemical reactions (electrochemical processes of cathodic protection), and the diffusion and convection of the constituent concentrations of various chemical substances, etc. Therefore, study of the numerical solution of the CDE is of great practical importance.[1−3]

At present, the studies on the numerical solution of the CDE have obtained many achievements: the lattice Boltzmann method has succeeded in numerically solving the Navier–Stokes equation of a fluid;[4] Liu *et al*[5] have put forward a lattice Boltzmann method for solving the two-dimensional CDE, thus taking one step closer to the solution of practical problems. Feng *et al*[6−11] had great success in using the retrospective multi-time-level integral scheme to solve the differential equations numerically. Later, based on the self-memory dynamics, Lu *et al*[12] derived the self-memory retrospective time integral scheme of a single parameter for the CDE. In recent years, due to the good local characteristics of the wavelet function, it has been successfully used in many fields; in particular, the discrete numerical algorithm of the wavelet has been widely used in the studies on many problems. Wang and Wei[13] constructed a QWNM based on the wavelet function to solve the Burgers equation, and

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the solution successfully describes the rapid change character of the function. Tang *et al*[14] obtained the numerical solution of the MKdV equation using the QWNM, and compared the solution with the analytical one, showing that the numerical solution has a high accuracy. We have solved numerically the CDE under the conditions of stochastically disturbed boundary and parameters by using the central explicit scheme, the improved central explicit scheme, the exponential type scheme, and the up-wind scheme, respectively, and examined the performances of four schemes. The results showed that the up-wind scheme is more accurate than the other three, and less affected by the boundary and parameter disturbances.[15,16]

On the basis of the previous works, we introduce the QWNM to solve the CDE, compare the solution with that obtained by using the up-wind scheme, investigate the effect of boundary and parameter disturbances on the solution obtained using the QWNM, and further discuss the range of computational bandwidth value W for the quasi-scale function expansion. The results indicate that W has an extremum, and when the value of W equals the extremum, the solution of the equation by using the QWNM possesses higher accuracy, and is better than that of the up-wind scheme, thus greatly reducing the computational efforts in large numerical prediction models. Especially, the QWNM is of great importance in large spectral models for saving computation time, and thus of merit in application.

2. Wavelet and quasi-wavelet

Since the last century, the wavelet has been an outcome of the important disciplines of modern harmonic analysis including function space, generalized function, Fourier analysis, and abstract harmonic analysis, and has been called a "mathematical microscope".[17] The major feature of a wavelet is its good local characteristics, i.e. its capability of fully highlighting some aspects of research problems after being transformed; therefore the wavelet is successfully used in many fields. The wavelet, as so called, is intuitively referred to as the shortest and simplest oscillations which can be observed by people. Mathematically, the wavelet is a function family,

$$h_{(a,b)}(x)=|a|^{-1/2}\,h\left(\frac{x-b}{a}\right)\quad(a,b\in R,a\neq 0),\tag{1}$$

obtained by the translation and stretch/shrink of such a function $h(x)$ in the function space $L^2(R)$ so that $h(x)$ satisfies

$$\int_{\mathrm{R}}h(x)\mathrm{d}x=0.\tag{2}$$

Therefore, the wavelet is sometimes also called the wavelet generating function. Equation (2) is a permissible condition showing that function $h(x)$ possesses a wave character. The waves of wavelet generating function $h(x)$ deviate from the horizontal axis only in the vicinity of the origin; when it is off the origin, the value of $h(x)$ decays rapidly to zero, and the entire wave approaches the horizontal. This is the basic reason why the function $h(x)$ is called "wavelet". For an arbitrary parameter pair (a,b), obviously we have

$$\int_{\mathrm{R}}h_{(a,b)}(x)\mathrm{d}x=0.$$

However, function $h_{(a,b)}(x)$ exhibits obvious disturbances only in the vicinity of $x=b$, and the range of the disturbances depends completely on the variation of parameter a. When $a>1$, the range is larger than that of the original wavelet generating function $h(x)$, and the waveform of the wavelet becomes shorter and thicker, and the change in the shape of the entire function becomes slow; when $0<a<1$, the range is smaller than that of the original one, and the waveform becomes sharper and thinner; when $a>0$ and a becomes smaller and smaller, the waveform of the wavelet gradually approaches an impulse function, and the shape of the function changes more and more rapidly. The regularities of change of wavelet function $h_{(a,b)}(x)$ with the parameter pair (a,b) determine that the wavelet function may have a good local characteristic, i.e. the ability to analyse the arbitrary sophisticated structure of the functions and the signals in an arbitrary assigned place. In order to further improve the local characteristics of the wavelet function, the scale function of the wavelet can be regularized. The regularized scale function is called the quasi-scale function, the wavelet generated through which is called the QWNM.[17]

3. Numerical computation

3.1. Construction of the difference form of the CDE

Consider the CDE

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=a\frac{\partial^2 u}{\partial x^2},x\in[0,2],t>0,\tag{3}$$

where a and c are constants, and $a>0$. The initial and boundary conditions of Eq.(3) in this paper are given as

$$u(x,0)=\sin\pi x, x\in[0,2];\quad u(0,t)=u(2,t)=0. \quad (4)$$

The spatial coordinate x is homogeneously discretized. The spatial step is taken as $2/N$, where N is the total number of grid units in [0,2], and the coordinates of grid points are $x_j=jh$, $j=1,2,\cdots,N$; u_j denotes the value of unknown function u at grid x_j. The time derivative in Eq.(3) is discretized by using the fourth-order Runge–Kutta method, and its discrete form is

$$y_j^{n+1}=y_j^n+\frac{\Delta t}{6}(k_{j,1}^n+2k_{j,2}^n+2k_{j,3}^n+k_{j,4}^n),$$
$$(j=1,2,\cdots,N+2), \quad (5)$$

where

$$\{y_j\}=\{y_1,y_2,y_3,\cdots,y_{N+2}\}=(t,u_0,u_1,\cdots,u_N),$$
$$(j=1,2,\cdots,N+2), \quad (6)$$

$$y_1^0=t=0,\quad y_j^0=u_{j-2}^0=\sin[\pi(j-2)h], \quad (7)$$

$$k_{1,1}^n=k_{1,2}^n=k_{1,3}^n=k_{1,4}^n=1. \quad (8)$$

Discretizing Eq.(3) by using the quasi-wavelet scheme[13,14] yields the following expressions of $k_{j,1}^n$, $k_{j,2}^n$, $k_{j,3}^n$, $k_{j,4}^n$ $(j=2,3,\cdots,N+1)$:

$$k_{j,1}^n=a\sum_{k=-w}^{w}\delta_{h,\sigma}^{(2)}(-kh)u_{j+k-2}^n - c\sum_{k=-w}^{w}\delta_{h,\sigma}^{(1)}(-kh)u_{j+k-2}^n, \quad (9)$$

$$k_{j,2}^n=a\sum_{k=-w}^{w}\delta_{h,\sigma}^{(2)}(-kh)\left(u_{j+k-2}^n+\frac{\Delta t}{2}k_{j+k,1}^n\right) - c\sum_{k=-w}^{w}\delta_{h,\sigma}^{(1)}(-kh)\left(u_{j+k-2}^n+\frac{\Delta t}{2}k_{j+k,1}^n\right), \quad (10)$$

$$k_{j,3}^n=a\sum_{k=-w}^{w}\delta_{h,\sigma}^{(2)}(-kh)\left(u_{j+k-2}^n+\frac{\Delta t}{2}k_{j+k,2}^n\right) - c\sum_{k=-w}^{w}\delta_{h,\sigma}^{(1)}(-kh)\left(u_{j+k-2}^n+\frac{\Delta t}{2}k_{j+k,2}^n\right), \quad (11)$$

$$k_{j,4}^n=a\sum_{k=-w}^{w}\delta_{h,\sigma}^{(2)}(-kh)\left(u_{j+k-2}^n+\Delta t k_{j+k,3}^n\right) - c\sum_{k=-w}^{w}\delta_{h,\sigma}^{(1)}(-kh)\left(u_{j+k-2}^n+\Delta t k_{j+k,3}^n\right). \quad (12)$$

The superscript n in the above expressions is the time level, and Δt represents the time step, $t=n\Delta t$ the integral time, and σ=3.2h a window parameter. $\delta_{h,\sigma}(x)$ is the regularized scale function, $\delta_{h,\sigma}^{(1)}(x)$ and $\delta_{h,\sigma}^{(2)}(x)$ are its first and second derivatives with respect to x, respectively. $[-W, W]$ is the computational bandwidth, and W is an integer. The derivation of Eqs.(4)–(10) and the computational formulae for the regularized scale function and its first and second derivatives can be found in Refs.[13] and [14].

The above computational process follows the below order: (i) calculating the four coefficients in Eqs.(9)–(12) by using the given initial value (i.e. Eqs.(7) and (8)) or the last time level value y_j^n; (ii) calculating the value of y_j^{n+1} $(j=1,2,\cdots,N+1)$ by using the iteration equation (5), and obtaining directly $y_2^n=y_{N+2}^n=0$ under the boundary condition (4); (iii) obtaining $y_j^{n+1}=u_{j-2}^{n+1}$ $(j=2,3,\cdots,N+2)$ from Eq.(6); (iv) by using the calculated y_j^{n+1} and the boundary condition, repeating the above computational procedure until obtaining the required time.

The difference equation of Eq.(3) obtained using the up-wind scheme is

$$u_j^{n+1}=u_j^n-\frac{c\Delta t}{h}(u_j^n-u_{j-1}^n) + \frac{a\Delta t}{h^2}(u_{j+1}^n-2u_j^n+u_{j-1}^n). \quad (13)$$

The performance of the QWNM in the convection–diffusion equation is measured with the following root-mean-square (RMS) error:

$$\mathrm{RMS}=\sqrt{\frac{\sum_{i=1}^{N}(x_i-\bar{x})}{N}}, \quad (14)$$

where $\bar{x}=\frac{1}{N}\sum_{i=1}^{N}x_i$, and x_i is the approximate solution of $u(x)$ at a certain time at the ith grid. If the RMS error for a method is smaller, then the method is better as far as this index is concerned.

3.2. The solution of CDE with QWNM

Taking the CDE as an example, the range of W in the expansion of quasi-scale function is emphatically discussed, the stability of the solution of the CDE by using QWNM is holohedrally examined under the four circumstances: the fixed boundaries, stochastic parameters, stochastic boundaries and stochastically disturbed initial values, and their solutions are compared with those obtained in the up-wind scheme in the following.

In the control numerical experiment (CTL) of fixed boundaries and parameters, the a, c, and N is given the values 0.1, 0.05, and 200, respectively, and the time step takes 0.0002, and $u(0)=u(N)=0.0$ is assumed. In the experiments of stochastic boundaries, the boundaries are disturbed by an equal-amplitude oscillation, i.e. $u(0)=u(N)=\mathrm{Ran}(P)/10.0$, and by oscillations of different amplitudes, i.e. $u(0)=\mathrm{Ran}(P)/10.0$ and $u(N)=\mathrm{Ran}(Q)/10.0$; parameters a and c have the same values as in the CTL; and Ran(P) and Ran(Q) are stochastic numbers between 0 and 1, and P and Q are stochastic seed (the same is kept below). In the experiments of stochastic parameters, when parameter a takes a stochastically disturbed value $a=0.05+(\mathrm{Ran}(P)/5.0\text{–}0.1)/5.0$, parameter c takes the same value as in the CTL. While parameter c takes a stochastically disturbed value $c=\mathrm{Ran}\ (P)/5.0$, parameter a takes the same value as in the CTL; under the conditions of the above two types of parameters, the boundary conditions are both the same as in the CTL.

When the initial conditions are stochastically disturbed, the initial values could be in the form $u(i) = \sin \pi x_i + \mathrm{Ran}(P)/100.0, x_i = i \times h, i = 1, 2, \cdots, N-1$; the boundary conditions and parameters a, c are the same as those in the fixed boundaries scheme.

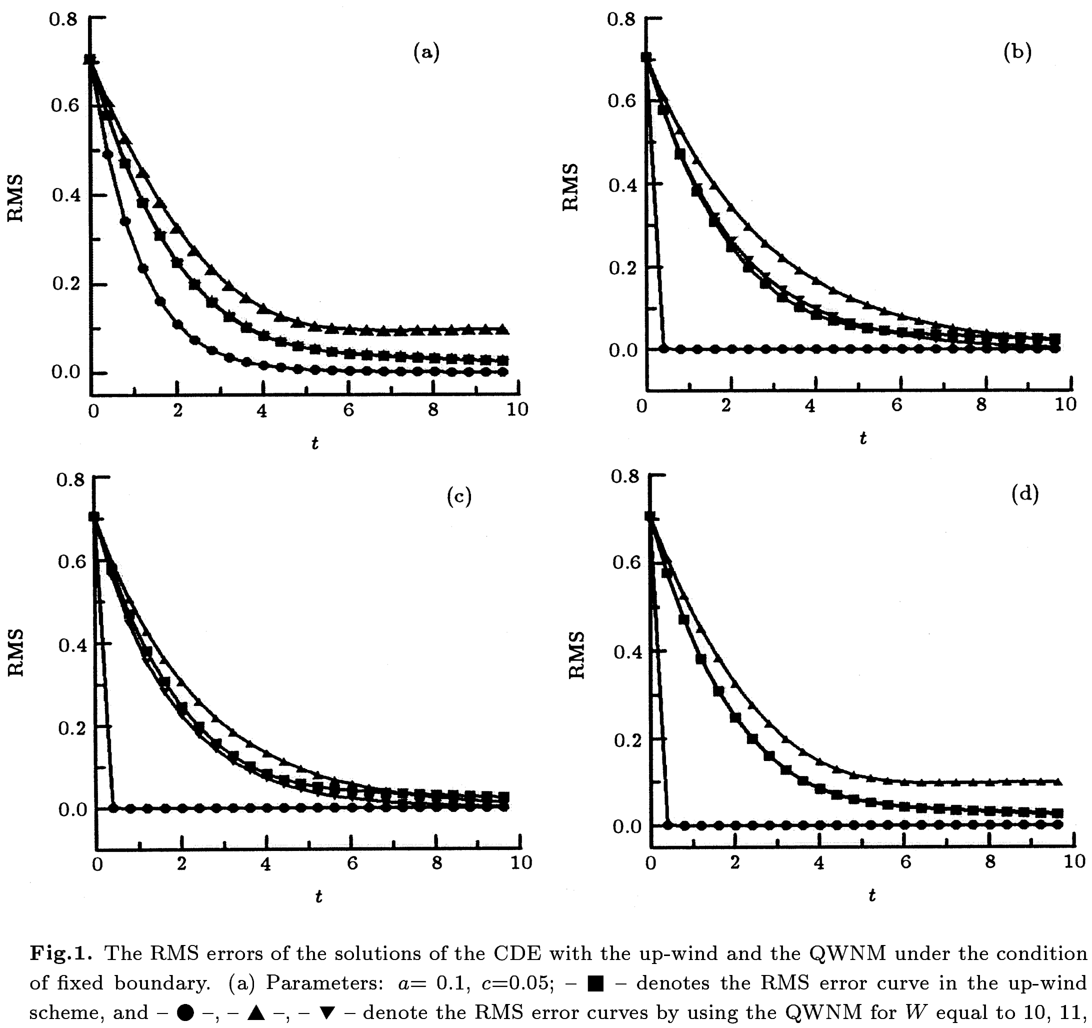

Fig.1. The RMS errors of the solutions of the CDE with the up-wind and the QWNM under the condition of fixed boundary. (a) Parameters: $a=0.1$, $c=0.05$; – ■ – denotes the RMS error curve in the up-wind scheme, and – ● –, – ▲ –, – ▼ – denote the RMS error curves by using the QWNM for W equal to 10, 11, and 20, respectively. (b) Parameters: $a=0.05+(\mathrm{Ran}(P)/5.0\text{–}0.1)/5.0$, $c=0.05$; – ● –, – ▲ –, – ▼ – the same as (a) but W equal to 6, 11, and 20, respectively. (c) Parameters: $a=0.1$, $c=\mathrm{Ran}\ (P)/5.0$; the others mean the same as those in (b). (d) Parameters: $a=0.1$, $c=0.05$; the others mean the same as those in (b).

It can be observed from Fig.1 (a) that for $W=10$, 11, and 20, the evolution of RMS errors of the solutions of the CDE with QWNM coincide with that of the up-wind scheme, i.e. the curves drop rapidly at first, and then approach the horizontal in the integral time-period. The RMS error curve of the solution with QWNM for $W=20$ is almost completely identical with that of the solution in the up-wind scheme, and the RMS error of the solution by using the QWNM for $W=10$ is obviously smaller than that of the up-wind scheme. It is interesting to note that, in contrast, the RMS error of the solution with QWNM for W=11 is larger than that of the up-wind scheme. The solutions with QWNM for W=12–40 (figures omitted) are also compared with those in the up-wind scheme, and the results show that the CDE can be well solved when W takes these values; however, the RMS errors of the solutions are all larger than those for W=10, and very close to or completely identical to the RMS error of the up-wind scheme. Furthermore, the solutions of Eq.(3) by using the QWNM for W being an integer greater than 20 are the same, but for W being an integer smaller than 9, the RMS errors of the solution of Eq.(3) by using the QWNM approaches infinity, and the solution is completely distorted.

Figures 1(b), 1(c) and 1(d) display the temporal evolution curves of the RMS errors of the solutions of Eq. (3) in the QWNM and the up-wind scheme when parameters a, c and the initial values are disturbed respectively. It can be seen from the figures that for W=6, the RMS error of the solutions of Eq.(3) by using the QWNM is the smallest, and it is almost zero in the integral time; for W=11, the RMS error of the solution is the largest, and greater than that of the up-wind scheme; and for W=20, the other results of RMS error are similar to those under the condition of fixed boundary and parameters.

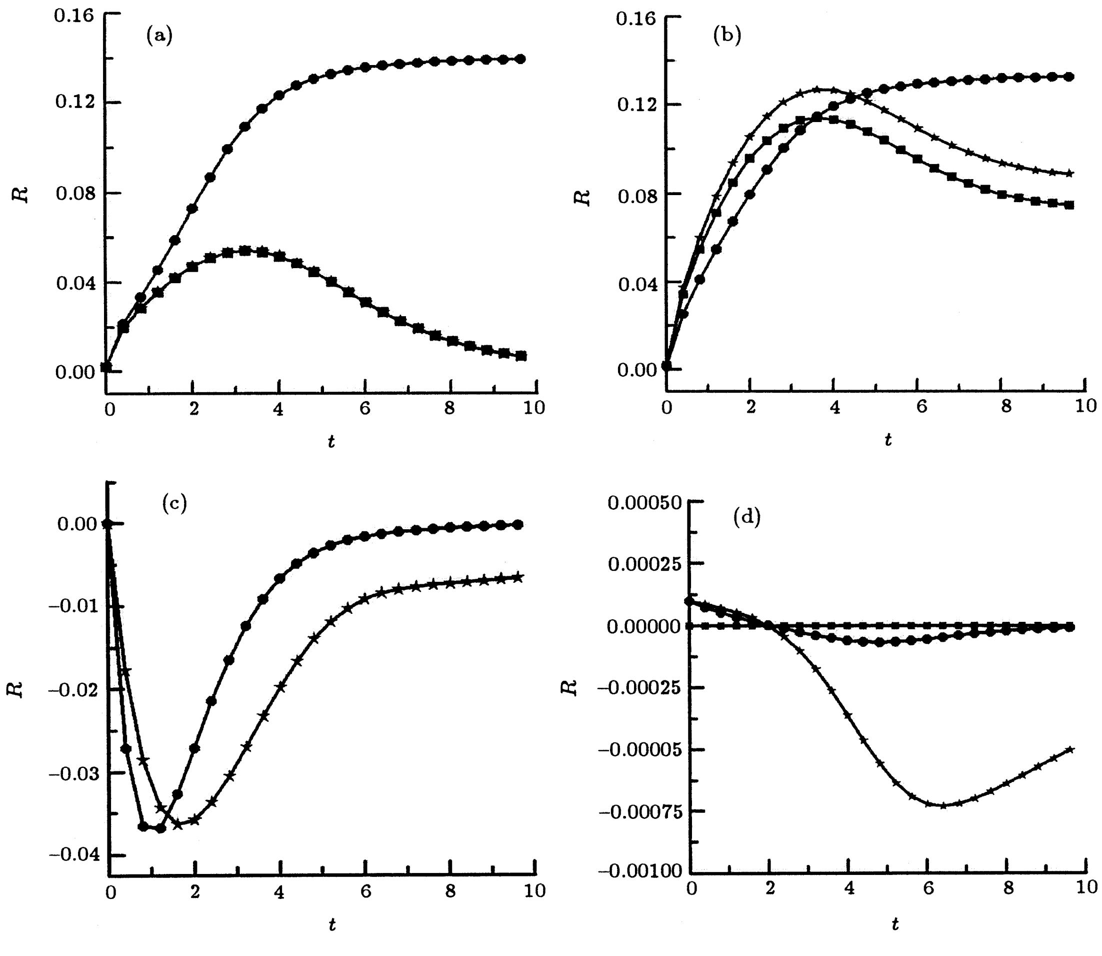

Fig.2. The differences (R) between the RMS errors of the solutions of the CDE by using the QWNM and the up-wind scheme, and that of the corresponding solutions in the CTL under the stochastically disturbed conditions. – ■ – means the same as in Fig.1; – ● –, – ★ – denote the R curves in the QWNM for W=10,20, respectively. (a) Equally disturbed boundaries; a=0.1, c=0.05; (b) Unequally disturbed boundaries; a=0.1, c=0.05; (c) Stochastically disturbed parameters; a=0.05+(Ran(P)/5.0–0.1)/5.0, c=0.05; (d) Disturbed initial values; a=0.1, c=0.05; $u(i) = \sin \pi x_i + \mathrm{Ran}(P)/100.0$, $x_i = i \times h, i = 1, 2, \cdots, N-1$.

Figure 2(a) shows that the R curve of the solution of Eq.(3) by using the QWNM for W=10 continuously rises with the increase of integral time, and the solution is strongly affected by disturbances, while the R curve for W=20 coincides completely with that of the up-wind scheme in the integral time period, and the curve rises at first and falls afterwards. It is worth noticing that in this circumstance the RMS error curve of the quasi-wavelet solution for W=20 still completely coincides with that of the up-wind scheme. However, the RMS error curve of the solution for W=10 is lower than that of the up-wind scheme before t=4.0, and afterwards it is higher than the curve until the end of integral time (figure omitted). Therefore, the solution of the CDE by using the QWNM for W=10 is more affected by boundary disturbances, while the solution is less affected for W=20.

Figure 2(b) is the time evolution of the R curves under the condition of unequally disturbed boundaries. The variation trend of the R curve of the solution for W=20 is similar to that of the up-wind scheme. The curve rises rapidly at first, and then falls slowly in the integral time period. It coincides almost with that of the up-wind scheme before t=0.8, and afterwards becomes always higher until the end of the integral period. The R curve of the solution for W=10 keeps increasing in the whole integral time period, obviously the solution being strongly affected by the disturbances. In this circumstance, the accuracy of the solution is higher than that of the solution with the QWNM for W=20 and the up-wind scheme before t=6.5, and in the whole integral time the accuracy of the solution with the up-wind scheme is slightly higher than that of the solution by using the QWNM for W=20 (Figure omitted).

Figure 2(c) displays the time evolution of the R curves when parameter a is disturbed. It can be seen that the R value of the solution by using the QWNM for W=20 is least affected by the parameter disturbances, its R curve almost completely coincides with that of the solution with the up-wind scheme, and the R value of the solution for W=10 is most affected by the parameter disturbances. Even so, its RMS error is still smaller than those of the solution for W=20 and the solution with the up-wind scheme (figure omitted). The conclusions obtained from the computation under the condition that the disturbed parameter c are similar to those under the condition of the disturbed parameter a. Because of limited space, we do not go into details.

Figure 2(d) is the time evolution of the R curves under the condition of disturbed initial values. The solution in the up-wind scheme is not influenced. Therefore, the solution of Eq.(3) by using the QWNM for W=20 is more affected by initial disturbances and less influenced for W=10. Even so, the accuracy of the solution of the CDE by using the QWNM for W=10 is relatively high, and better than that of the up-wind scheme (figure omitted).

4. Conclusions

This paper has introduced the QWNM to solve the CDE, and compared its computational results with those of the up-wind scheme. It is found that the QWNM can be used to solve the CDE satisfactorily. Conclusions are summarized as follows:

(i) When using the QWNM to solve the CDE, the calculated bandwidth W has an extremum. If the bandwidth W takes specific values, the accuracy of the solution of the CDE by using the QWNM is relatively high, and better than that of the up-wind scheme. While W takes an integer greater than or equal to 20, the various solutions are completely the same.

(ii) Under the condition of the boundary disturbances of equal amplitudes, the solution of Eq.(3) by using the QWNM is less affected for W=20, and the fluctuation amplitude of its R curve is the same as that of the up-wind scheme. Under the condition of the boundary disturbances of different amplitudes, the solutions of Eq.(3) by using up-wind scheme are least affected. Under the boundary disturbances, the solution is more influenced by using the QWNM for W=10, and with the increase of the integral time its accuracy gradually becomes lower than that of the up-wind scheme and the QWNM for W=20.

(iii) Under the condition of parameter disturbances, the solution by using the QWNM for W=20 coincides with that of the up-wind scheme and is most stable, while the solution for W=10 is most affected, but the accuracy is still higher than that of the QWNM for W=20 and the up-wind scheme.

(iv) Under the stochastically disturbed initial value, the accuracy of the solution by using the QWNM for W=10 in the CDE remains relatively high and better than that of the up-wind scheme.

It can be seen clearly that under the condition of parameter disturbances, although the solution by using the QWNM for W=10 is the most affected, the accuracy remains high, which is the same as the conclusion for the fixed boundaries. Under the condition of boundary disturbances, for a long integral time the accuracy for W=10 is lower than that of the QWNM for W=20 and the up-wind scheme. Because the solution by using QWNM for W=10 is more affected by the disturbances, in a long integral time, the numerical solution may be distorted. Therefore, in practical applications of the QWNM, it is necessary to properly select the computational bandwidth W according to different parameter, boundary and initial conditions so as to raise the accuracy of the numerical solution and reduce the influence of stochastic disturbances on the numerical solution.

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